# On the Propagation of Alfvén Waves in Certain Rotational Symmetric Magnetic Fields*

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The general results of previous work [1, 2] are applied to the propagation of ALFVÉN waves in certain rotational symmetric magnetic fields. It is shown that for finite field strength no bound states of such waves exist. The transmission and reflection coefficients are calculated for a certain class of simple fields.

In some special cases the problem can be solved rigorously, in other cases it is convenient to apply BORN approximation.

**I.** It has been shown in a previous paper [2] that the solution of the ALFVÉN wave equation for a wave propagating in an inhomogeneous field can be reduced to the solution of an ordinary differential equation if the following conditions are satisfied:

1. In the undisturbed equilibrium configuration electric current-lines and magnetic field-lines shall be directed perpendicularly everywhere.
2. The velocity vector of the disturbed configuration shall be directed parallel to the current lines.
3. Nothing shall depend on a curvilinear coordinate directed parallel to the current lines.

A general curvilinear coordinate system was introduced. In this coordinate system one family of coordinate lines is directed along the field lines, the second family of coordinate lines along the electric current lines and the third family of coordinate lines perpendicular to both of it.

The ALFVÉN wave equation under these conditions can then be brought into the form:

$$\varrho_0(\omega)^2 v^2 + u^1 g^{22} (g_{22} u^1 v^2{}_{,1})_{,1} = 0. \tag{1.1}$$

Upper indices indicate contravariant and lower indices covariant tensor components. The comma sign stands for ordinary differentiation.

In (1.1) are $\varrho_0$ the equilibrium density, in general space dependent, $\omega$ the frequency of the wave. $\boldsymbol{v}$ is the velocity vector with the contravariant components

$$v^\alpha = \{0, v^2, 0\}.$$

$\boldsymbol{u}$ is a vector connected with the magnetic field vector $\boldsymbol{H}_0$ of the equilibrium configuration by the equation:

$$\boldsymbol{u} = \boldsymbol{H}_0 / \sqrt{4\pi}.$$

The contravariant components of $\boldsymbol{u}$ are given by

$$u^\alpha = \{u^1, 0, 0\}.$$

$g_{ik}$ is the metric tensor being diagonal and having the components:

$$g_{ik} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix}.$$

Equation (1.1) must be supplemented by the equation of continuity. Since ALFVÉN waves are associated with no change in density this equation is:

$$(\sqrt{g}\, v^2)_{,2} = 0. \tag{1.2}$$

A third equation being of importance is the divergence equation of the magnetic field $\boldsymbol{H}_0$ resp. $\boldsymbol{u}$. This equation is given by:

$$(\sqrt{g}\, u^1)_{,1} = 0. \tag{1.3}$$

Since nothing shall depend on $x^2$ equation (1.2) is automatically satisfied.

**II.** Of special importance with regard to practical applications are ALFVÉN waves propagating in fields possessing rotational symmetry.

In this case the coordinate line $x^2$ is directed along the azimuth $\varphi$. The two other coordinate lines the $x^1$ and $x^3$ lines are located in the $r$, $z$ plane of a cylindrical coordinate system, $r$, $\varphi$, $z$.

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[1] R. GAJEWSKI and F. WINTERBERG, Boeing Scientific Res. Rep., No. D 1-82-0111 and Annals of Physics, New York, to be published.

[2] F. WINTERBERG, Z. Naturforschg. 18 a, 545 [1963].

The metric tensor has the general form:

$$g_{ik} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & g_{33} \end{pmatrix} \tag{2.1}$$

and the wave equation (1.1) reduces to

$$\varrho_0 (\omega)^2 v^2 + u^1 \cdot (r)^{-2} ((r)^2 \cdot u^1 v^2{}_{,1})_{,1} = 0. \tag{2.2}$$

The derivatives in (2.2) are taken with regard to the coordinate $x^1$. It is however, more expedient to take the derivative in an invariant way that is independent of the scaling of the coordinate values $x^1$. For this reason we express the derivatives with regard to $x^1$ by derivatives with regard to the arc length $s$ along the coordinate line $x^1$. This connection for any function $F$ is given by

$$F_{,1} = \frac{\mathrm{d}F}{\mathrm{d}x^1} = \frac{\mathrm{d}F}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}x^1} = \sqrt{g_{11}}\frac{\mathrm{d}F}{\mathrm{d}s} = \sqrt{g_{11}}\,F'. \tag{2.3}$$

The upper prime shall indicate derivation with regard to the arc length $s$ of $x^1$. Introducing the derivative with regard to the arc length $s$ in (2.2) we obtain

$$\varrho_0 (\omega)^2 v^2 + \sqrt{g_{11}}\, u^1 \cdot (r)^{-2} ((r)^2 \cdot \sqrt{g_{11}}\, u^1 v^{2\prime}) = 0. \tag{2.4}$$

Next we introduce in (2.4) instead of the contravariant component $u^1$ the "physical component" $u$ related to it by

$$u = \sqrt{g_{11}}\, u^1 \tag{2.5}$$

with the result upon (2.4):

$$\varrho_0 (\omega)^2 v^2 + u \cdot (r)^{-2} ((r)^2 \cdot u\, v^{2\prime})' = 0. \tag{2.6}$$

The physical component $v$ of the contravariant component $v^2$ is given similarly by

$$v = \sqrt{g_{22}}\, v^2 = r\, v^2. \tag{2.7}$$

From (2.7) it is evident that the contravariant component $v^2$ is just the angular velocity of the torsional ALFVÉN wave. We substitute therefore instead of $v^2$ the angular velocity $\Omega$. Thus equation (2.6) has finally the form

$$\varrho_0\, \omega^2\, \Omega + u\, r^{-2} (u\, r^2\, \Omega')' = 0. \tag{2.6a}$$

**III.** For any given field configuration $r$ and $u$ are known functions of position and thus equation (2.6a) can be always solved. On this basis however, it is not possible to obtain some general results which can be easily compared with experiments. For each field configuration it is necessary to carry out a cumbersome numerical calculation, first to determine $u$ and $r$ as a function of position then to solve (2.6a).

It is however possible to simplify the solution of (2.6a) by observing that the torsional wave propagates along the field lines and thus along magnetic flux tubes.

If we approximate this flux $\Phi_{\mathrm{m}}$ by the expression

$$\Phi_{\mathrm{m}} \cong \pi\, r^2\, H_0 \propto u\, r^2 = \text{const} \quad (\text{along } x^1) \tag{3.1}$$

we can simplify equation (2.6a) into

$$\varrho_0\, \omega^2\, \Omega + u^2\, \Omega'' = 0 \tag{3.2}$$

or

$$\Omega'' + k^2(s)\, \Omega = 0 \tag{3.3}$$

with

$$k^2(s) = \omega^2 / a^2(s) \tag{3.4}$$

and

$$a = H_0 / \sqrt{4\pi\, \varrho_0} \tag{3.5}$$

the ALFVÉN speed.

Equation (3.3) can now be solved with standard procedures if $a^2(s)$ and thus $k^2(s)$ is a known function of the arc length along the field lines [3].

For certain magnetic fields however, equation (3.1) is not approximately but rigorously satisfied.

To explore the properties of such fields we rewrite equation (3.1) in the form

$$(u\, r^2)_{,1} = 0\,. \tag{3.6}$$

A second equation, the magnetic field has to satisfy, is the divergence condition (1.3). By substituting in it the "physical" component of $u$ according to (2.5) and with

$$g = g_{11}\, g_{22}\, g_{33} = g_{11}\, g_{33}\, r^2$$

it can be brought into:

$$(\sqrt{g_{33}}\, r\, u)_{,1} = 0\,. \tag{3.7}$$

Integrating (3.6) and (3.7) yields the two equations:

$$u\, r^2 = f_1(x)\,, \tag{3.8}$$

$$\sqrt{g_{33}}\, r\, u = f_2(x^3)\,. \tag{3.9}$$

$f_1(x^3)$, $f_2(x^3)$ are arbitrary functions of $x^3$. Eliminating $u$ from (3.8) and (3.9) results in [$f_3(x^3)$, $f(x^3)$ again arbitrary functions of $x^3$]

$$r = \sqrt{g_{33}}\, f_3(x^3) \tag{3.10}$$

resp.

$$g_{33} = r^2\, f(x^3)\,. \tag{3.11}$$

[3] The approximation for the magnetic flux given by equ. (3.1) made in connection with the adiabatic invariance of the magnetic moment of a particle leads to the conlusion that a particle moves on flux tubes.

The current distribution comparable with magnetic fields of this character is easily calculated. The current-density $j$ follows from MAXWELL's equation:

$$j = (c/\sqrt{4\pi}) \operatorname{curl} \boldsymbol{u} \tag{3.12}$$

taking the covariant expression for the curl which has only in the $x^2$ direction a nonvanishing component:

$$\operatorname{curl} u\,|^2 = g^{-1/2} \varepsilon^{2\beta\gamma} u_{\gamma;\beta}$$
$$= g^{-1/2} u_{1,3} = g^{-1/2} (g_{11} u^1)_{,3} = g^{-1/2} (\sqrt{g_{11}}\, u)_{,3} \tag{3.13}$$

(semicolon stands for the covariant derivative).

One thus obtains for the covariant component of the current density

$$j^2 = (c/\sqrt{4\pi})\, g^{-1/2} (\sqrt{g_{11}}\, u)_{,3} \tag{3.14}$$

or by making use of (3.8), (3.11) and introducing the physical component for $j^2$: $j = \sqrt{g_{22}}\, j^2 = r\, j^2$:

$$j = \frac{c}{\sqrt{4\pi}} \frac{1}{r\sqrt{g_{11} f(x^3)}} (\sqrt{g_{11}}\, u)_{,3} \tag{3.15a}$$

$$= \frac{c}{\sqrt{4\pi}} \frac{1}{r\sqrt{g_{11} f(x^3)}} \frac{\sqrt{g_{11}}\, f_1(x^3)}{r^2}{}_{,3}\,. \tag{3.15b}$$

This calculation shall be supplemented by a geometrical consideration to understand the meaning of equation (3.11). For this purpose we consider at some distance $z = z_0$ of our cylindrical coordinate system $r$, $\Phi$, $z$ two field lines at $r = r_0$ and $r = r_0 + \mathrm{d}r$ (Fig. 1). Now let us assume that the slope of two

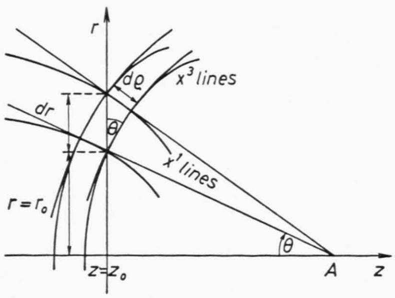


Fig. 1.

neighboring field lines possess a common vertex which is just located on the axis of symmetry, say at $z = A$. It is then easy to see that if the field lines have this property, they obey condition (3.11). To show this we set up a polar coordinate system, $\varrho$, $\theta$ the origin of which is located at the vertex point $A$. This polar coordinate system will then locally coincide with the curvilinear coordinate system of the field lines and its orthogonal trajectories at the point $r_0$, $z_0$. The line element in the polar coordinate system is given by the expression

$$\mathrm{d}s^2 = \mathrm{d}\varrho^2 + \varrho^2\, \mathrm{d}\theta^2. \tag{3.16}$$

Now it is $\quad \varrho = \dfrac{r_0}{\sin\theta}, \qquad \mathrm{d}\varrho = \sin\theta\, \mathrm{d}r \qquad (3.17)$

therefore $\quad \mathrm{d}s^2 = \mathrm{d}\varrho^2 + \dfrac{r_0^2}{\sin^2\theta}\, \mathrm{d}\theta^2. \qquad (3.18)$

Now locally we can put:

$$\varrho = x^1, \qquad \theta = x^3$$

from this follows that $g_{33}$ has the desired form:

$$g_{33} = r^2 f(x^3)\,.$$

For geometrical reasons, also, it is apparent that the field line structure, as indicated in Fig. 1, must obey the condition:

$$u \propto 1/r^2 \text{ (along } x^1).$$

All field lines located in the $r$, $z$ plane obey then the parametric equation $r = \alpha(x^3)\, f(z)$. $\alpha(x^3)$ varies from field line to field line and $f(z)$ is a function common to all field lines.

Fig. 2 – 4 show examples for such fields.

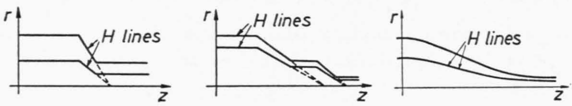


Fig. 2. Fig. 3. Fig. 4.

It should be pointed out that there is no restriction on the choice of $u(s)$. Any $u(s)$ which is a nonsingular function is permissible. This can be seen immediately. Any field line has to obey the equation

$$u\, r^2 = C \tag{3.19}$$

where $C$ is some parameter changing from field line to field line. Since $u = u(s)$ it follows from (3.19) that $r = r(s)$ and therefore also $s = s(r)$ is for a given $u = u(s)$ a known function. From $s = s(r)$ follows by differentiation with respect to $z$:

$$\frac{\mathrm{d}s}{\mathrm{d}z} = \sqrt{1 + \left(\frac{\mathrm{d}s}{\mathrm{d}r}\right)^2} = \frac{\mathrm{d}s}{\mathrm{d}r} \cdot \frac{\mathrm{d}r}{\mathrm{d}z} \tag{3.20}$$

and from this by integration

$$z - z_0 = \int \frac{(\mathrm{d}s/\mathrm{d}r)}{\sqrt{1 + (\mathrm{d}s/\mathrm{d}r)^2}}\, \mathrm{d}r\,. \tag{3.21}$$

$(\mathrm{d}s/\mathrm{d}r)$ is a given function of $r$, therefore for any given $u = u(s)$ and given field line parameter $C$ the equation of the field line can be determined by a quadrature.

But even if any function $k^2(s)$ is permissible, it might be difficult however, to set up the necessary current distribution to realize the corresponding field.

It is especially suggestive to select for $k^2(s)$ such functions in which equation (3.3) is solvable in a closed form.

**IV.** It might be suspected that for the calculation of reflection and transmission coefficients of ALFVÉN waves by field inhomogeneties only the width and height of a "magnetic potential barrier" is the predominate parameter not the associated distribution of the electric currents necessary to create the field and which might be quite complicated.

In order to study the reflection and transmission of an ALFVÉN wave let us first apply BORN approximation. For this we start from (3.3) which we write in a different form:

$$\Omega'' + (k_\infty{}^2 - U(s))\,\Omega = 0 \tag{4.1}$$

with
$$k_\infty{}^2 = \omega^2/a_\infty{}^2 \tag{4.2}$$

and
$$U(s) = \omega^2\left(\frac{1}{a_\infty{}^2} - \frac{1}{a^2(s)}\right). \tag{4.3}$$

$a(s)$ is the ALFVÉN velocity at any position $s$ whereas $a_\infty$ is the same velocity at $\pm\infty$.

Next we approximate $U(s)$ by a GAUSSian with the half width $\Gamma$

$$U(s) = C\exp\{-\ln 2\cdot(s/\Gamma)^2\}. \tag{4.4}$$

In order to be compatible with (4.3) one obtains for the constant $C$:

$$C = \frac{1}{a_\infty{}^2} - \frac{1}{a_0{}^2}, \tag{4.5}$$

$a_0$ is the ALFVÉN velocity at $s=0$.

Therefore

$$\begin{aligned} U(s) &= k_\infty{}^2\left(1-\left(\frac{a_\infty}{a_0}\right)^2\right)\exp\{-\ln 2\cdot(s/\Gamma)^2\} \\ &= k_\infty{}^2\left(1-\left(\frac{H_\infty}{H_0}\right)^2\right)\exp\{-\ln 2\cdot(s/\Gamma)^2\}. \end{aligned} \tag{4.6}$$

$H_0$ and $H_\infty$ are the magnetic field strengths at $s=0$, $\pm\infty$.

The reflection-coefficient is given by the expression:

$$R = \frac{1}{4\,k_\infty{}^2}\left|\int_{-\infty}^{+\infty}\exp\{i\,k_\infty\,s\}\,U(s)\,\Omega(s)\,\mathrm{d}s\right|^2. \tag{4.7}$$

The transmission coefficient $T$ can be calculated from:

$$R + T = 1. \tag{4.8}$$

In first BORN approximation we put

$$\Omega(s) = \exp\{i\,k_\infty\,s\} \tag{4.9}$$

and obtain from (4.7)

$$\begin{aligned} R &= \frac{1}{4}\,k_\infty{}^2\left(1-\left(\frac{H_\infty}{H_0}\right)^2\right)^2 \cdot\left|\int_{-\infty}^{+\infty}\exp\{-\ln 2\cdot(s/\Gamma)^2 + 2\,i\,k_\infty\,s\}\,\mathrm{d}s\right|^2 \\ &= \frac{\pi(\Gamma\,k_\infty)^2}{4\ln 2}\exp\left\{-\frac{2}{\ln 2}(k_\infty\,\Gamma)^2\right\}\cdot\left(1-\left(\frac{H_\infty}{H_0}\right)^2\right)^2. \end{aligned} \tag{4.10}$$

Equation (4.10) is a simple expression from which the reflection and transmission of an ALFVÉN wave can be easily calculated.

**V.** In some cases it is possible to solve equation (3.3) rigorously. Equation (3.3) is solvable for the function $k^2(s)$ which is given by

$$k^2(s) = k_\infty{}^2 + (k_0{}^2 - k_\infty{}^2)\left(1-\tanh^2\frac{s}{\Gamma}\right). \tag{5.1}$$

(5.1) has just the property that the wave number $k^2(\pm\infty) = k_\infty{}^2$ for $s=\pm\infty$ and $k^2(0) = k_0{}^2$ for $s=0$. Depending on whether $k_0{}^2 \lesseqgtr k_\infty{}^2$, (5.1) represents a "well" or a "barrier". $\Gamma$ is the "width" of the "potential". It is convenient to introduce

$$(k_0/k_\infty)^2 = (H_\infty/H_0)^2 \tag{5.2}$$

where $H_0$ and $H_\infty$ are the magnetic field strengths at $s=0$, $\pm\infty$. One obtains then instead of (5.1)

$$k^2(s) = k_\infty{}^2\left[1-\left(1-\left(\frac{H_\infty}{H_0}\right)^2\right)\left(1-\tanh^2\frac{s}{\Gamma}\right)\right]. \tag{5.3}$$

For $k^2(s)$ given by (5.1) resp. (5.3) it is well known [4] that the reflection coefficient resp. transmission coefficient is given by:

$$R = \frac{1+\cos\pi\beta}{\cosh(2\,\pi\,\Gamma\,k_\infty)+\cos\pi\beta} = 1 - T, \tag{5.4}$$

$$\beta = (1 + (2\,k_\infty\,\Gamma)^2(1-(H_\infty/H_0)^2))^{1/2}. \tag{5.5}$$

(5.4) is also valid if $\beta$ is imaginary. For

$$1 + (2\,k_\infty\,\Gamma)^2(1-(H_\infty/H_0)^2) > 0$$

the reflection coefficient is a periodic function of $\beta$. On the other hand if

$$1 + (2\,k_\infty\,\Gamma)^2(1-(H_\infty/H_0)^2) < 0$$

the reflection coefficient is a hyperbolic function of $i\,\beta$ which is a hyperbolic function of real argument since $\beta$ will be imaginary.

[4] P. MORSE and H. FESHBACH, Methods of Theoretical Physics, McGraw-Hill, New York 1953, p. 1657.

In summary:

$$\left(\frac{H_\infty}{H_0}\right)^2 \lessgtr 1 + \frac{1}{(2\,k_\infty\,\Gamma)^2} \begin{matrix}\text{periodic}\\ \text{unperiodic}\end{matrix} \;. \qquad (5.6)$$

Another solvable example[5] for $k^2(s)$ is the "smooth step" function

$$k^2(s) = k_-^2 + \frac{1}{2}\,(k_+^2 - k_-^2)\left(1 + \tanh\frac{s}{\Gamma}\right) \qquad (5.7)$$

From (5.7) follows that

$$k^2(-\infty) = k_-^2, \qquad k^2(+\infty) = k_+^2\,.$$

We note that

$$(k_-/k_+)^2 = (H_+/H_-)^2$$

in which $H_+$, $H_-$ are the magnetic field strength at $s = \pm\infty$. The reflection coefficient for a wave coming in from $s = -\infty$ is then given by:

$$R = \left[\frac{\sinh[(\pi/2)\,\Gamma\,k_+(1-(H_-/H_+))]}{\sinh[(\pi/2)\,\Gamma\,k_-(1+(H_-/H_+))]}\right]^2. \qquad (5.8)$$

In the limiting case if $\Gamma k_- \ll 1$ (5.8) becomes asymptotically

$$R \to [(H_+ - H_-)/(H_+ + H_-)]^2. \qquad (5.9)$$

From (5.8) and (5.9) it is clear that in order to make $R=1$

$$\text{either} \quad H_-/H_+ = 0 \quad \text{or} \quad H_-/H_+ = \infty\,.$$

In the first case $H_-/H_+ = 0$ total reflection is possible only if the wave propagates into a region in which the magnetic field goes to infinity. In the second case $H_-/H_+ = \infty$ total reflection is only possible if the wave propagates into a region in which the field goes to zero.

[5] S. Flügge and H. Marschall, Rechenmethoden der Quantenmechanik, Springer-Verlag, Berlin 1952, p. 75.

# Die Beeinflussung der Fluoreszenz von Molekülen durch ein äußeres elektrisches Feld. I. Theorie

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Es wird eine Theorie der Beeinflussung der Fluoreszenz von Molekülen durch ein äußeres elektrisches Feld entwickelt. Die Theorie ist auf Moleküle mit oder ohne permanentes Dipolmoment und mit isotroper oder anisotroper Polarisierbarkeit anwendbar. Die Umorientierung der fluoreszenzfähigen angeregten Moleküle wird als kinetisches Phänomen behandelt. Eine explizite Rechnung wird für das Modell eines hinreichend starren Moleküls durchgeführt. Die Anwendung der Gleichung gestattet es, bei geeigneten Molekülen das Dipolmoment des angeregten Zustandes zu bestimmen.

Durch Bestrahlung mit Licht einer geeigneten Frequenz kann ein Molekül unter Absorption eines Lichtquants aus dem Elektronengrund- in einen Anregungszustand überführt werden. Die Übergangswahrscheinlichkeit eines Moleküls hängt außer von der Intensität und der Frequenz des einfallenden Lichtes von der Orientierung des Moleküls relativ zum elektrischen Vektor der Lichtwellen ab. In einer Lösung sind die Moleküle im allgemeinen über alle Orientierungen isotrop verteilt, und die mittlere Übergangswahrscheinlichkeit wird bei Bestrahlung mit linear polarisiertem Licht unabhängig von der Polarisationsrichtung.

In einem äußeren elektrischen Feld wird die Energie eines Moleküls mit einem permanenten Dipolmoment oder mit anisotroper Polarisierbarkeit von der Orientierung des Moleküls zum äußeren Feld abhängig. Dies verursacht eine Anisotropie der Orientierungsverteilung der gelösten Moleküle. Die Lösungen werden dichroitisch, d. h. die mittlere Übergangswahrscheinlichkeit und damit der Extinktionskoeffizient hängen von der Polarisationsrichtung der einfallenden Lichtwelle relativ zum äußeren elektrischen Feld ab. Ist das Dipolmoment (permanentes Dipolmoment plus induziertes Dipolmoment) eines Moleküls im Grundzustand dem Betrag oder der Richtung nach von dem Moment des Anregungszustandes verschieden, dann werden die Energieniveaus des Moleküls im Grund- und Anregungszustand im elektrischen Feld um verschiedene Beträge verändert. Daher wird die Anregungsenergie zwischen zwei bestimmten Zuständen im elektrischen